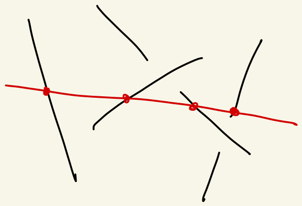
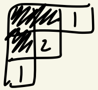
Flag Varieties & Schubert Calculus

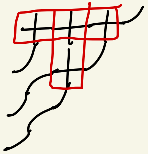
Part I

Jake Levinson

Simon Fraser University

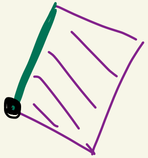


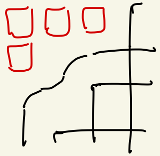


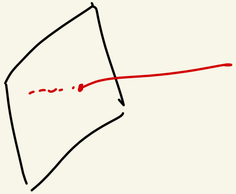


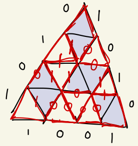
$\mathfrak{S}_w$

$H^*(G/B)$









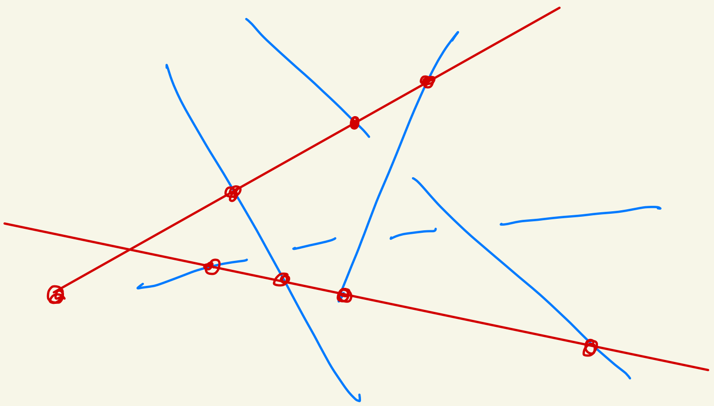
Combinatorics Days in Covilhã

July 2023, Portugal

Geometry of planes

Q1. How many solutions does $\underset{\underset{m \times n}{\nearrow}}{A}\,\underset{\underset{\in \mathbb{R}^n}{\uparrow}}{\vec{x}} = \underset{\underset{\in \mathbb{R}^m}{\nwarrow}}{\vec{b}}$ have? 0, 1 or ∞.

Q2. Given 4 general lines in $\mathbb{C}^4$, how many lines intersect all 4? $\underline{\underline{2}}$.



Schubert calculus answers questions about linear algebra using combinatorics.

geometry of planes, vector spaces, flags

representation theory, algebra, polynomials

partitions, permutations, tableaux, words, puzzles...

Ex: Q2 becomes:

How many ways can we fill in $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$ by $\{1, 2, 3, 4\}$

such that entries increase $\uparrow\!\rightarrow$? Ans: $\begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array}$, $\begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array}$. Two.

Subspaces and Flags

A k-dim'l vector subspace $V \subseteq \mathbb{C}^n$ can be described as the row span of a k×n matrix:

$$V = \text{row}\left(\begin{array}{c} \uparrow \\ k \\ \downarrow \end{array} \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kn} \end{bmatrix} \right).$$

That is, $\underset{\subseteq \mathbb{C}^{kn}}{\{\text{full rank } k\times n \text{ matrices}\}} \xrightarrow{\text{row span}} \!\!\!\!\to \{k\text{-dim'l subspaces } V \subseteq \mathbb{C}^n\} := Gr(k,n).$

This is called the Grassmannian,

The entries x_{ij} are called Stiefel coordinates and they are not unique:

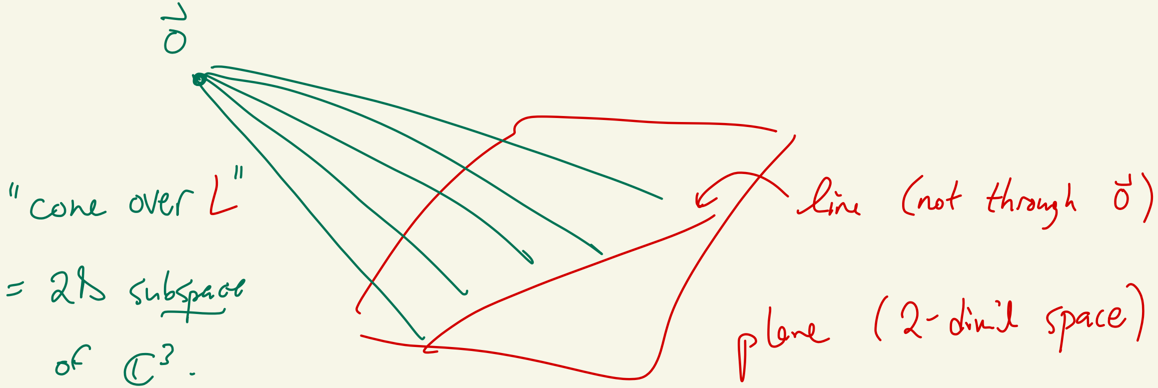
$$\text{e.g.} \quad \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \underset{\text{row ops}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & -1/4 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/4 \end{bmatrix}$$

$$\text{row}(A) = \text{row}(g \cdot A), \quad g \in GL_k.$$

left-multiply ↔ row operations.

Fact: $Gr(k,n) = {}_{GL_k}\backslash \{\text{full-rank } k \times n \text{ matrices}\}$

Rmk: k-dim'l subspace of $\mathbb{C}^n$ $\longleftrightarrow$ $(k-1)$-plane (not through the origin) in $\mathbb{P}^{n-1}$.



Running example: $Gr(2,4) = \{\text{dim } 2 \text{ subspaces in } \mathbb{C}^4\} = \{\text{lines in } \mathbb{P}^3\}$.

(3D space)

Schubert cells and varieties

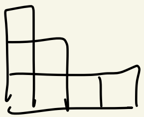
Recall: Every matrix is row-equivalent to a unique matrix in (row) echelon form:

e.g.
$$\begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

with 0's left of pivots and in pivot columns.

pivot columns $247 = \underbrace{(123)}_{\text{leftmost possible}} + (124)$

$\lambda = 124 =$



a Young diagram or integer partition.

You can check: $\det_{247}$ is the leftmost nonzero maximal minor.

(unchanged by row operations).

Note: most matrices reduce to the form

$$\left[\begin{array}{cccc} 1 & 0 & 0 & * \cdots\cdots * \\ 0 & 1 & 0 & * \qquad\quad * \\ 0 & 0 & 1 & \underbrace{* \cdots\cdots *}_{n-k} \end{array}\right] \Big\updownarrow k \qquad \begin{array}{l} \lambda = (0\,0\,0) \\ \text{number of stars} = k(n-k) \end{array}$$

$(\Leftrightarrow \det_{123} \neq 0)$.

We call $X^\circ_\lambda = \{ \text{subspaces } V \in Gr(k,n) : \det_{(1\cdots k)+\lambda} \text{ is } \underline{\text{leftmost} \neq 0} \det \}$,

a (Grassmannian) Schubert cell.

The closure $X_\lambda = \overline{X^\circ_\lambda}$ is a (Grassmannian) Schubert variety.

Facts:

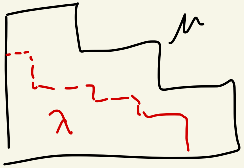
- $\mathrm{Gr}(k,n) = \coprod_{\lambda \subseteq \underset{k \times (n-k)}{\square}} X^\circ_\lambda$ complex cell structure on $\mathrm{Gr}(k,n)$.

- $X^\circ_\lambda \simeq \mathbb{C}^{k(n-k) - |\lambda|}$, $|\lambda| = \#\mathrm{boxes}(\lambda)$

  That is, $\mathrm{codim}\left(X^\circ_\lambda\right) = |\lambda|$.

- $X_\lambda = \coprod_{\lambda \subseteq \mu} X^\circ_\mu$

Why we index with partitions!



Ex: $Gr(2,4) = \{2 \text{ dim'l subspaces of } \mathbb{C}^4\} \longleftrightarrow \{\text{lines } \mathbb{P}^1 \text{ in } \mathbb{P}^3 \text{ projective space}\}$.

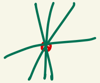
Schubert stratification:

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/x & 1 & 0 \end{bmatrix} \right)$$

As $x \to \infty$

$$\begin{tikzcd}
 & & \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \arrow[dr, dash] & & \\
\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \arrow[r, dash] & \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \arrow[ur, dash] \arrow[dr, dash] & & \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \arrow[r, dash] & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 & & \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \arrow[ur, dash] & &
\end{tikzcd}$$

lines containing a certain point



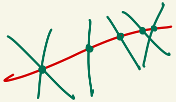
$\lambda = \square\square$

$\lambda = \emptyset$

general lines

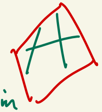
$\lambda = \square$

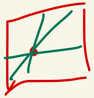
lines meeting a certain line



$\lambda = \begin{smallmatrix}\square\\\square\end{smallmatrix}$

lines in a certain plane $\mathbb{P}^2 \subseteq \mathbb{P}^3$





$\lambda = \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$

$\|$

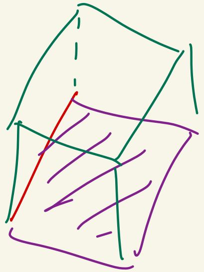
$X_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \cap X_{\square\square}$

$\lambda = \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}$

a specific line!

One point of $Gr(2,4)$.

As the example suggests, a Schubert variety prescribes how the k-plane meets certain other planes.



Def: A complete flag $\mathcal{F}_\bullet$ in $\mathbb{C}^n$ is a sequence of subspaces

$$F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n, \quad \dim F_i = i.$$

Def: $X_\lambda(\mathcal{F}_\bullet) = \{ \text{subspaces } V \text{ such that } \dim(V \cap \mathcal{F}_i) \boxed{\geq} n-k+i-\lambda_i \}$

"=" for $X^\circ_\lambda(\mathcal{F}_\bullet)$

Fact: "Echelon form" = X_λ using the "standard reverse flag":

$$\mathcal{F}_\bullet = \underset{F_1}{\langle e_n \rangle} \subseteq \underset{F_2}{\langle e_n, e_{n-1} \rangle} \subseteq \cdots \subseteq \langle e_n, \ldots, e_1 \rangle = \mathbb{C}^n .$$

Crash course: Cohomology and Counting

The cohomology ring $H^*(Gr(k,n))$ is a graded ring that lets us count solutions to geometry problems.

Elements: equivalence classes of formal sums of closed subvarieties

e.g. $X_\lambda \subseteq Gr(k,n) \rightsquigarrow$ cohomology class $[X_\lambda]$.

if X deforms continuously to Y in $Gr(k,n) \rightsquigarrow [X] = [Y]$.

Ring structure: $+$: formal sum / union

$\cdot$: If $X, Y \subseteq Gr(k,n)$ intersect transversely,

$$[X] \cdot [Y] = [X \cap Y].$$

← enables enumerative geometry!

Ex: $\{\text{lines through 4 given lines } L_1, L_2, L_3, L_4\} \subseteq Gr(2,4)$

$$= \underset{[X_{\square}]}{\{\text{lines through } L_1\}} \cap \underset{\cdot\ [X_{\square}]}{\{\text{lines through } L_2\}} \cap \underset{\cdot\ [X_{\square}]}{\{\cdots L_3\}} \cap \underset{\cdot\ [X_{\square}]}{\{\cdots L_4\}}$$

$= [X_{\square}]^4$ (this is transverse if $L_1, \dots, L_4$ are general).

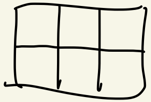
Pieri's rule: $\lambda \cdot \square = \sum \lambda'$ obtained by adding a box to λ (staying in $k\,\underbrace{\boxed{\qquad\quad}}_{n-k}$.)

$$\underbrace{\square \cdot \square} \cdot \square \cdot \square = \left(\ydiagram{1,1} + \ydiagram{2}\right) \cdot \square \cdot \square$$

$$= \left(\ydiagram{1,2} + \ydiagram{1,2}\right) \cdot \square \qquad \ydiagram{1,1,1}\ \ \ydiagram{3} \not\subseteq \ydiagram{2,2}$$

$$= 2\ \ydiagram{2,2} \leftarrow \text{cohomology class of a point.} = 2 \text{ points in } Gr(2,4).$$

Ex: Calculate $\square^6$ in $H^*(Gr(3,6))$.



Ans: 5 points.

$$\begin{matrix}4&5&6\\1&2&3\end{matrix}\qquad\begin{matrix}3&5&6\\1&2&4\end{matrix}\qquad\begin{matrix}3&4&6\\1&2&5\end{matrix}\qquad\begin{matrix}2&5&6\\1&3&4\end{matrix}\qquad\begin{matrix}2&4&6\\1&3&5\end{matrix}$$

Fact: $$H^*(Gr(k,n)) = \bigoplus_{\lambda \subseteq k\underset{n-k}{\square}} \mathbb{Z}\cdot[X_\lambda]$$

⊛ Every class in H^* is equiv. to a sum of $[X_\lambda]$'s.

In particular, $$[X_\lambda]\cdot[X_\mu] = \sum_{\substack{\nu:\\ |\nu| = |\lambda|+|\mu|,\\ \nu \subseteq \square}} c_{\lambda\mu}^{\nu}\,[X_\nu]$$

(H^* is graded by codimension)

for some structure constants $c_{\lambda\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$.

counting certain intersections!

Jewel of combinatorics: Explicit combinatorial formulas for $c_{\lambda\mu}^{\nu}$.

Schur polynomials

These theorems are shadows of a polynomial model for $H^*(Gr(k,n))$:

Theorem: There is a surjective ring map

$$\mathbb{Z}[x_1,\dots,x_k]^{S_k} \twoheadrightarrow H^*(Gr(k,n)),$$

Ring of symmetric polynomials

sending the Schur polynomial $s_\lambda(x_1,\dots,x_k) \longmapsto [X_\lambda]$.

Many formulas exist for the Schur polynomial, the most-used one is as follows.

Def: A *semistandard Young tableau* T of shape λ is a filling of λ by natural numbers, such that rows *weakly* increase → $\boxed{2\,2}$ ok

cols *strictly* increase ↑ $\cancel{\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}}$ not ok.

The *weight* of T is $x^T := x_1^{\#1\text{'s}} x_2^{\#2\text{'s}} \cdots$.

Ex:

$$\begin{array}{llll} 4 & & & \\ 2 & 2 & & \\ 1 & 1 & 2 & 3 \end{array}$$

Weight $x^T = x_1^2 x_2^3 x_3 x_4$.

Def: $$s_\lambda(x_1,\ldots,x_k) = \sum_{\substack{T \text{ semistandard,} \\ \text{shape } \lambda,\ \text{entries} \le k}} x^T .$$

Ex: $$s_{\begin{array}{|c|c}\hline \ & \\ \hline \ & \multicolumn{1}{c|}{\ } \\ \hline\end{array}}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 .$$

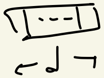
$$\begin{array}{ll}2 & \\ 1 & 1\end{array} \quad \begin{array}{ll}3 & \\ 1 & 1\end{array} \quad \begin{array}{ll}2 & \\ 1 & 2\end{array} \quad \begin{array}{ll}3 & \\ 1 & 2\end{array} \quad \begin{array}{ll}2 & \\ 1 & 3\end{array} \quad \begin{array}{ll}3 & \\ 1 & 3\end{array} \quad \begin{array}{ll}3 & \\ 2 & 3\end{array}$$

Ex: $$s_{d\updownarrow\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}(x_1,\ldots,x_k) = \text{the } d^{\text{th}} \text{ elementary symmetric polynomial}$$

$$= \sum_{i_1<i_2<\cdots<i_d\le k} x_{i_1}x_{i_2}\cdots x_{i_d} .$$

Multiplying

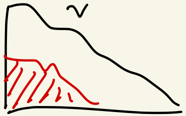
Pieri rule: $s_\lambda \cdot s$  $= \sum s_{\lambda'}$ over all λ' obtained by adding d boxes to λ in different rows

$s_\lambda \cdot s$  $= \sum s_{\lambda'}$ - - - - - - - - - - different columns

Ex: 

⊛⊛ **Littlewood-Richardson rule:** $\lambda \cdot \mu = \sum_{|\nu|=|\lambda|+|\mu|} c^{\nu}_{\lambda\mu} \cdot \nu$

where $c^{\nu}_{\lambda\mu} = \#\Big\{$ ballot tableaux of shape ν/λ



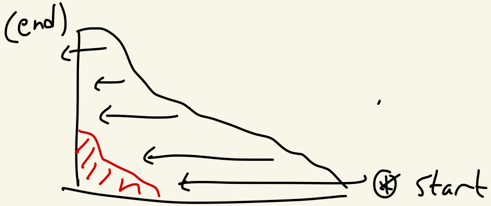
(delete λ)

of weight $\mu \Big\}$

$\{\mu_1$ 1's, μ_2 2's, etc.

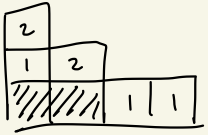
Def: We say a semistandard tableau T is ballot or L-R if:

Read the entries:



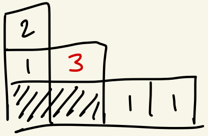
At every step, we require $\#1\text{'s} \geq \#2\text{'s} \geq \#3\text{'s} \geq \cdots$.

Ex:



ballot

11212 ✓



not ballot

11312

Ex: You try: